IB DIPLOMA PROGRAMME
PROGRAMME DU DIPLÔME DU BI
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M07/5/MATHL/HP2/ENG/TZ2/XX

22077208

MATHEMATICS
HIGHER LEVEL
PAPER 2

Tuesday 8 May 2007 (morning)

2 hours

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Maximum mark: 27]*

Consider the vectors $\boldsymbol{a} = \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k}$ and $\boldsymbol{c} = 2\boldsymbol{i} - 5\boldsymbol{j} - \boldsymbol{k}$.

(a) Given that $\boldsymbol{c} = m\boldsymbol{a} + n\boldsymbol{b}$ where $m, n \in \mathbb{Z}$, find the value of m and of n. *[5 marks]*

(b) Find a unit vector, $\boldsymbol{u}$, normal to both $\boldsymbol{a}$ and $\boldsymbol{b}$. *[5 marks]*

(c) The plane π_1 contains the point A$(1, –1, 1)$ and is normal to $\boldsymbol{b}$. The plane intersects the x, y and z axes at the points L, M and N respectively.

(i) Find a Cartesian equation of π_1.

(ii) Write down the coordinates of L, M and N. *[5 marks]*

(d) The line through the origin, O, normal to π_1 meets π_1 at the point P.

(i) Find the coordinates of P.

(ii) **Hence** find the distance of π_1 from the origin. *[7 marks]*

(e) The plane π_2 has equation $x + 2y + 4z = 4$. Calculate the angle between π_2 and a line parallel to $\boldsymbol{a}$. *[5 marks]*

2. *[Total Mark: 21]*

Part A *[Maximum mark: 11]*

The times taken for buses travelling between two towns are normally distributed with a mean of 35 minutes and standard deviation of 7 minutes.

(a) Find the probability that a randomly chosen bus completes the journey in less than 40 minutes. *[2 marks]*

(b) 90 % of buses complete the journey in less than t minutes. Find the value of t. *[5 marks]*

(c) A random sample of 10 buses had their travel time between the two towns recorded. Find the probability that exactly 6 of these buses complete the journey in less than 40 minutes. *[4 marks]*

Part B *[Maximum mark: 10]*

The number of bus accidents that occur in a given period of time has a Poisson distribution with a mean of 0.6 accidents per day.

(a) Find the probability that at least two accidents occur on a randomly chosen day. *[4 marks]*

(b) Find the most likely number of accidents occurring on a randomly chosen day. Justify your answer. *[3 marks]*

(c) Find the probability that no accidents occur during a randomly chosen seven-day week. *[3 marks]*

3. *[Maximum mark: 22]*

Let $\boldsymbol{A} = \begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix}$.

(a) Find the values of λ for which the matrix $(\boldsymbol{A} - \lambda \boldsymbol{I})$ is singular. *[5 marks]*

Let $\boldsymbol{A}^2 + m\boldsymbol{A} + n\boldsymbol{I} = \boldsymbol{O}$ where $m, n \in \mathbb{Z}$ and $\boldsymbol{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

(b) (i) Find the value of m and of n.

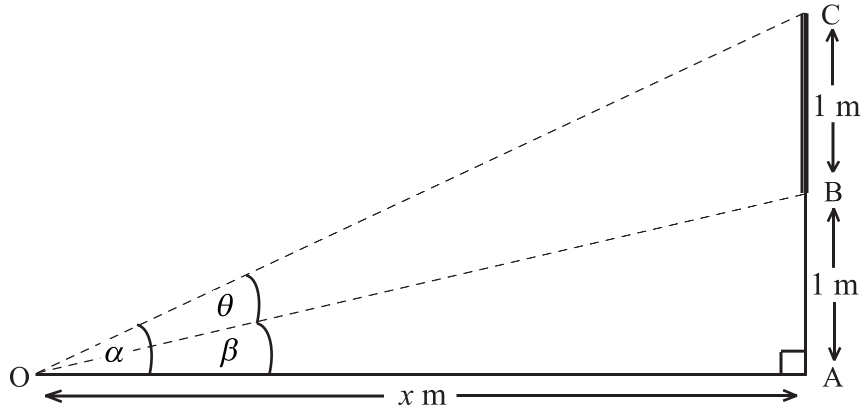
(ii) Hence show that $\boldsymbol{I} = \frac{1}{5}\boldsymbol{A}(6\boldsymbol{I} - \boldsymbol{A})$.

(iii) Use the result from **part (b) (ii)** to explain why $\boldsymbol{A}$ is non-singular. *[12 marks]*

(c) Use the values from **part (b) (i)** to express $\boldsymbol{A}^4$ in the form $p\boldsymbol{A} + q\boldsymbol{I}$ where $p, q \in \mathbb{Z}$. *[5 marks]*

4. *[Maximum mark: 22]*

A television screen, BC, of height one metre, is built into a wall. The bottom of the television screen at B is one metre above an observer's eye level. The angles of elevation ($\text{A}\hat{\text{O}}\text{C}$, $\text{A}\hat{\text{O}}\text{B}$) from the observer's eye at O to the top and bottom of the television screen are α and β radians respectively. The horizontal distance from the observer's eye to the wall containing the television screen is x metres. The observer's angle of vision ($\text{B}\hat{\text{O}}\text{C}$) is θ radians, as shown below.



(a) (i) Show that $\theta = \arctan\frac{2}{x} - \arctan\frac{1}{x}$.

(ii) Hence, or otherwise, find the **exact** value of x for which θ is a maximum and justify that this value of x gives the maximum value of θ.

(iii) Find the maximum value of θ. *[17 marks]*

(b) Find where the observer should stand so that the angle of vision is 15°. *[5 marks]*

5. *[Maximum mark: 28]*

Let $u = 1 + \sqrt{3}\mathrm{i}$ and $v = 1 + \mathrm{i}$ where $\mathrm{i}^2 = -1$.

(a) (i) Show that $\frac{u}{v} = \frac{\sqrt{3}+1}{2} + \frac{\sqrt{3}-1}{2}\mathrm{i}$.

(ii) By expressing both u and v in modulus-argument form show that $\frac{u}{v} = \sqrt{2}\left(\cos\frac{\pi}{12} + \mathrm{i}\sin\frac{\pi}{12}\right)$.

(iii) **Hence** find the **exact** value of $\tan\frac{\pi}{12}$ in the form $a + b\sqrt{3}$ where $a, b \in \mathbb{Z}$. *[15 marks]*

(b) Use mathematical induction to prove that for $n \in \mathbb{Z}^+$,
$\left(1+\sqrt{3}\mathrm{i}\right)^n = 2^n\left(\cos\frac{n\pi}{3} + \mathrm{i}\sin\frac{n\pi}{3}\right)$. *[7 marks]*

(c) Let $z = \frac{\sqrt{2}v + u}{\sqrt{2}v - u}$.

Show that $\mathrm{Re}\, z = 0$. *[6 marks]*